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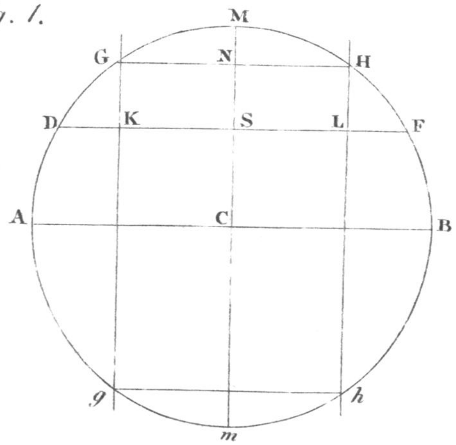

Fig. 1.

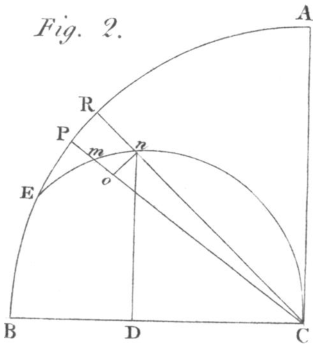

Fig. 2.

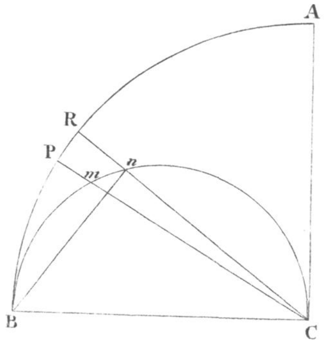

Fig. 3.

On determining INNUMERABLE PORTIONS *of a* SPHERE, *the* SOLIDITIES *and* SPHERICAL SUPERFICIES *of which* PORTIONS *are at the ſame Time algebraically aſſignable.*

By the Rev. J. BRINKLEY *A. M. ANDREWS Profeſſor of Aſtronomy in the Univerſity of Dublin.*

IN the year 1692, Viviani, a Florentine, who had been the pupil of Galileo, propoſed to the mathematicians of the age a problem, in which it was required to aſſign a portion of the ſurface of a ſphere exactly quadrible. This problem, truly indeterminate, and known by the name of the Florentine problem, was readily ſolved by ſeveral in various ways*. The conſtruction, however, of Viviani, which was afterwards publiſhed by himſelf, without demonſtration, exceeded the others in ſimplicity and elegance. It cannot now be known in what manner Viviani **Read, Nov. 2, 1801.**

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* Wallis's Works, vol. ii. p. 478, and the Leipſic Acts, 1692.

Viviani demonſtrated his conſtruction. It may be readily and accurately done by methods familiar to geometricians before the diſcovery of fluxions and of the differential calculus. Yet Viviani ſays of the problem, "Cujus divinatio, a ſecretis artibus "illuſtrium analyſtarum vigentis ævi, expectatur, quod in geometriæ "puræ hiſtoria tantummodo verſatus ad tam recondita videatur inva-"lidus." By "ſecretæ artes" there can be no doubt he deſigned the method of fluxions and the differential calculus (at that time in manner concealed by Newton and Leibnitz); it therefore may be ſuppoſed that Viviani's demonſtration was not the ſimpleſt one of which the conſtruction of his problem admits. Undoubtedly the demonſtration, independent of fluxions, is much leſs difficult than the problems propoſed by Paſcal concerning the cycloidal ſolids, in which the mathematicians of the middle of the ſeventeenth century were ſo much engaged. Euler long afterwards propoſed to determine a portion of a ſphere, which inſtead of its ſurface ſhould have its ſolidity accurately aſſignable. Euler has given a very ingenious ſolution of this indeterminate problem, but not one that merits the epithet of elegant, which he juſtly gives to the problem. His ſolution, being derived from the method of coordinates and double integrals, is much leſs ſimple than what a conſideration of the properties of the ſphere would have furniſhed. By proceeding upon too general principles the eaſieſt caſe eſcaped him. This caſe has lately been diſcovered and announced in the ſecond mathematical volume of the French National Inſtitute, by M. Boſſut, from which it appears that Viviani's conſtruction of the Florentine problem determines at the ſame time a portion of the ſurface of the ſphere, and a portion of its ſolidity, both exactly aſſignable. M. Boſſut tells us, that the analyſis of his ſolution is much more difficult than that of Viviani's conſtruction, and promiſes to communicate it hereafter. In the mean time, Mr. Woodhouſe has given a very ingenious demonſtration of it in the firſt part of the Phil. Tranſ. for 1801. This demonſtration, however, being performed by the method of coordinates and double integrals, is leſs ſimple than the problem admits.

WHAT ſeems accidental in Viviani's conſtruction will take place in an innumerable variety of inſtances; and it is the object of the preſent Eſſay to ſhew that "*there are innumerable conſtructions by which portions of a ſphere may be obtained ſo that the ſpherical ſuperficies and ſolidity of each portion are accurately aſſignable.*

LEMMA.

LEMMA.

IF a ſphere be perforated by a right cylinder, the axis of which paſſes through the centre of the ſphere, the ſolid content of the remainder of the ſphere is to $\frac{1}{6}$ of a cube, the ſide of which is the length of the cylinder entirely within the ſphere, as the circumference of a circle to its diameter.

DEM. Let $AMBm$ be a ſection of the ſphere through the axis of the cylinder Mm. Let $GMHhmg$ be a ſection of the cylinder, $Gg = Hh$ its length wholly within the ſphere, and let AB and DF be drawn parallel to GH, gh which will therefore be diameters of ſections of the ſphere perpendicular to Mm. Fig. 1.

THE area of the ſection through DF : ſect. of cylind. :: DS^2 : KS^2 therefore dividendo

ſect. of remainder : ſect. of cylind. :: $DS^2 - KS^2$: KS^2
but ſect. of cylind. : great circle :: KS^2 : AC^2
whence

ſect. of remainder : great circle :: $DS^2 - KS^2$ (GN^2) = $CN^2 - CS^2$: AC^2; or ſect of remaind. : $CN^2 - CS^2$ :: great circle : AC^2 : circumf. : diameter.

Now if on the ſquare on CN as a baſe, a cube and pyramid of the ſame height be conſtructed, the difference of the areas of any ſections of the cube and pyramid at the diſtance CS from the vertex $= CN^2 - CS^2$. And the ſolids of equal heights, whoſe ſections at the ſame height are always in the ſame given ratio, are in the ratio of their baſes, therefore the remainder of hemiſphere : diff. between the cube and pyramid :: circumference : diameter. But the difference between the cube and pyramid $= \frac{2}{3}$ of the cube. Hence the remainder of the hemiſphere : $\frac{2}{3}$ cube on CN :: circumference : diameter, or remainder of ſphere : $\frac{4}{3}$ cube on CN $= \frac{1}{6}$ cube on Gg :: circumference : diameter. Q.E.D.

Problem. ABC being a quadrant of a great circle of a ſphere, the centre of which is C, it is required to find the nature of the curve $EmnC$, ſo that the ſurface of the ſphere inſiſting perpendicularly over the area $EAnmC$ may be algebraically aſſignable; and alſo that the ſolidity inſiſting perpendicularly on the ſame area may be algebraically aſſignable. Fig. 2.

Solution.

Fig. 2.

Solution. Let the rad. $AC = r$, $Cn = z$, the meaſure of the $\angle ACR$ to rad. unity $= a$, and let on be a circular arc deſcribed with radius Cn (z). Becauſe the ſurface of a zone contained between a great circle and a parallel circle equals the rectangle contained by the circumference of the great circle, and the perpendicular diſtance between the circles*, it readily follows that the ſurface of the ſphere inſiſting perpendicularly above and below $PRno = 2PR \times \sqrt{CR^2 - Cn^2} = 2PR \times \sqrt{r^2 - z^2}$. This ſurface is to the ſurface ſtanding perpendicularly on $PmnR$ ultimately in the ratio of equality. Therefore the fluxion of the ſurface over $ACnR = 2r\dot{a}\sqrt{r^2 - z^2}$. Again it eaſily follows from the lemma that the ſolid inſiſting perpendicularly on $PRno = \frac{PR}{2CR} \times \frac{1}{6}$ cube of the perp. at $n = \frac{2}{3}\frac{PR}{CR} \times \overline{r^2 - z^2}|^{\frac{3}{2}}$, and the ratio of this ſolid to the ſolid inſiſting on $PRnm$ is ultimately the ratio of equality. Hence the fluxion of the ſolid inſiſting on $ACnR = \frac{2}{3}\dot{a}\,\overline{r^2 - z^2}|^{\frac{3}{2}}$.

It follows therefore that the value of $\dot{a}$ muſt be ſuch that the fluents of $\left.\begin{matrix} 2r\dot{a}\sqrt{r^2 - z^2} \\ \frac{2}{3}\dot{a} \times \overline{r^2 - z^2}|^{\frac{3}{2}} \end{matrix}\right\}$ may be both algebraically expreſſed. This it readily appears can take place by aſſuming an infinite variety of values of $\dot{a}$ in terms of $\dot{z}$, z and conſt. quantities. The problem conſequently is truly unlimited.

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* Archimedes de Sphæra & Cylindro.

To conſider more particularly ſome of theſe caſes,

1 Let $\dot{a} = \frac{\dot{z}}{\sqrt{r^2 - z^2}}$

and the fluent of $\begin{cases} 2r\dot{a}\sqrt{r^2-z^2} = \text{fl. } 2r\dot{z} = 2rz = \text{(when } z = r\text{) } 2r^2 \\ \frac{2}{3}\dot{a}\times\overline{r^2-z^2}\big|^{\frac{3}{2}} = \text{fl. } \frac{2}{3}\dot{z}\times\overline{r^2-z^2} = \frac{2}{3}r^2z - \frac{2}{9}z^3 = \text{(when } z=r\text{)} \frac{4}{9}r^3. \end{cases}$

Becauſe in this hypotheſis $\dot{a} = \frac{\dot{z}}{\sqrt{r^2 - z^2}}$ or $\dot{a}\,r = \frac{r\dot{z}}{\sqrt{r^2 - z^2}}$ Fig. 3.

therefore $a\,r$ = circ. arc rad. r and ſine z, or s, A R = C n, or coſine of B R = C n Conſequently the angle B n C is a right angle, and the locus of n or the curve C $n\,m$ is a ſemicircle deſcribed on the diam. B C. This caſe therefore furniſhes the ſolution which Viviani himſelf gave of the Florentine problem.

If on the radius which is the continuation of B C another ſemicircle be deſcribed and the ſame conſtruction made, the ſurface of the hemiſphere inſiſting perpendicularly between the ſemicircle and the quadrants $= 2 \times 2\,r^2 =$ the ſquare of the diameter of the ſphere.

Also, if a circle be deſcribed on each radius of a diameter of a ſphere, and the ſphere be perforated by cylinders inſiſting perpendicularly on theſe circles, the remainder of the ſphere $= 4 \times \frac{4}{9} r3 = \frac{2}{9}$ cube on the diameter of the ſphere, as M. Boſſut has announced in the ſecond volume of the French National Inſtitute*.

2. If $\dot{a} = \frac{m z \dot{z}}{r^2}$ (*m* being a numerical coefficient) then the correct fluents of

$$\begin{cases} 2 r \dot{a} \times \overline{r^2 - z^2}\big\rangle^{\frac{1}{2}} = \frac{2}{3} m \times r^2 - \overline{\frac{r^2 - z^2}{r}}\Big\rangle^{\frac{3}{2}} \\ \frac{2}{3} \dot{a} \times \overline{r^2 - z^2}\big\rangle^{\frac{3}{2}} = \frac{2}{15} m \times r3 - \overline{\frac{r^2 - z^2}{r^2}}\Big\rangle^{\frac{5}{2}} \end{cases}$$

when $z = r$ the ſurface becomes $\frac{2}{3} m r^2$ and the ſolid $\frac{2}{15} m r3$. The curve in this caſe ſatisfying the problem cannot paſs through Fig. 2. B, for by taking the fluents $a = \frac{m z^2}{2 r^2}$, and therefore if the curve paſſed through B, $m = 2a =$ ſemi-circumference of a circle, the rad. of which is unity, and not an aſſignable number, conſequently the ſolid and ſurface could not be algebraically expreſſed.

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*By help of the theorem of Archimedes for the ſurface of a ſpherical zone Viviani's concluſion may, without difficulty, be accurately deduced by method known to geometricians before the invention of fluxions and by the help of the lemma above given Boſſut's theorem might alſo be deduced by the ſame method, independent of fluxions. Hence Viviani's aſſertion reſpecting the difficulty of ſolving the problem from geometrical conſideration alone; cannot be conſidered as juſt.

If *m* be taken, any number having a leſs ratio to unity than the circumference of a circle to its diameter the curve will cut A B. If *m* be taken, having a greater ratio to unity than the circumference to the diameter, the curve will cut the continuation of A B, and in both caſes the conditions of the problem will be anſwered.

If $\dot{a} = \frac{m z^{n} \dot{z}}{r^{n+1}}$, *n* being any odd number and *m* a numerical coefficient

$$\text{the fluents of}\ \begin{cases} 2 r \dot{a} \sqrt{r^2 - z^2} = \text{fl.}\ \dfrac{2 m z^{n} \dot{z}}{r^{n}} \sqrt{r^2 - z^2} \\ \dfrac{2}{3} \dot{a} \times \overline{r^2 - z^2}\big|^{\frac{1}{2}} = \text{fl.}\ \dfrac{2}{3} m z^{n} \dot{z} \times \overline{r^2 - z^2}\big|^{\frac{3}{2}} \end{cases}$$

will be had in finite terms, and therefore the conditions of the problem will be anſwered. From the equation $a = \frac{m z^{n+1}}{\overline{n+1}\, r^{n+1}}$ the value of *m* may be determined ſo that the curve may cut AB or its continuation. Becauſe $a = \frac{m z^{n} \dot{z}}{r^{n+1}}$, if p = the perpendicular from C on the tangent to the curve, $\frac{m z^{n+1} \dot{z}}{r^{n+1}} : \dot{z} :: p : \sqrt{z^2 - p^2}$. Therefore $r^{2n+2} p^2 = m^2 z^{2n+4} - m^2 z^{2n+2} p^2$ or $p = \frac{m z^{n+2}}{\sqrt{r^{2n+2} + m^2 z^{2n+2}}}$, the equation of the ſpiral of which E n C is a portion.

3. Let $\dot{a} = \frac{m z^{n} \dot{z}}{r^{n} \sqrt{r^2 - z^2}}$ n being any whole poſitive number, and m a numerical coefficient,

$$\text{the fluent of} \left\{ \begin{array}{l} 2 r \dot{a} \sqrt{r^2 - z^2} = \frac{2 m r^2}{n+1} \\ \frac{2}{3} \dot{a} \times \overline{r^2 - z^2}|^{\frac{3}{2}} = \frac{4 m r^3}{3\, \overline{n+1} \cdot \overline{n+3}} \end{array} \right\} \text{when } z = r.$$

Becauſe in this hypotheſis $\dot{a} = \frac{m z^{n} \dot{z}}{r^{n} \sqrt{r^2 - z^2}}$, when n is an odd number the relation of a and z will be algebraical; when

n is even the fluent of $\frac{m z^{n} \dot{z}}{r^{n} \sqrt{r^2 - z^2}}$ can be found by help of the circle.

Therefore if the value m be taken according to the different values of n ſo that z may become equal to r, innumerable ſolutions will be had.

By theſe and other obvious methods innumerable ſolutions may be obtained. Curves may be obtained that ſhall paſs through neither B nor C, or through B and not through C.

The curves above obtained are, except the circle, ſpirals.

4. If it be required that the curves be ſuch that the relation between the abſciſſa C D $= x$, and the ordinate D$n = y$ may be Fig. 2. algebraical; ſuch curves may be found, but not ſo readily as the above. When the curve is algebraical z is an algebraical function of x and y, whence, becauſe $z = \frac{x}{s,\, a}$ and $z = \frac{y}{cs,\, a}$, it follows that s, a and cs, a are algebraical functions of z.

1. Let $s, a = \frac{z}{r}$, then $cs, a = \sqrt{1 - \frac{z^2}{r^2}}$

therefore $\dot{a} = \frac{\dot{z}}{r} \times \frac{1}{\sqrt{1 - \frac{z^2}{r^2}}} = \frac{\dot{z}}{\sqrt{r^2 - z^2}}$

This value of a, as was ſhewn above, anſwers the conditions of the problem.

Now $x = z\,s, a = \frac{z^2}{r}, y = z\,cs, a = \sqrt{z^2 - \frac{z^4}{r^2}}$

whence $y^2 = z^2 - \frac{z^4}{r^2} = r\,x - x^2$, which is the equat. to the circle, as before found.

2. Let $s, a = \frac{z^2}{r^2}$, then $cs, a = \sqrt{1 - \frac{z^4}{r^4}}$

and $a = \frac{2\,z\,z}{r^2\sqrt{1 - \frac{z^4}{r^4}}} = \frac{2\,z\,z}{\sqrt{r^4 - z^4}}$

Hence $2\,r\,\dot{a}\sqrt{r^2 - z^2} = \frac{4\,z\,z}{\sqrt{r^2 + z^2}}$

$$\frac{2}{3}\,\dot{a} \times \overline{r^2 - z^2}|^{\frac{3}{2}} = \frac{4\,z\,\dot{z}}{\sqrt{r^2 - z^2}} \times \overline{r^2 - z^2}$$

conſequently the fluent of $\left\{\begin{array}{l} 2\,r\,\dot{a}\sqrt{r^2 - z^2} = \sqrt{2} - 1 \times 4\,r^2 \\ \frac{2}{3}\,\dot{a} \times \overline{r^2 - z^2}|^{\frac{3}{2}} = \frac{4}{9}\,r^3 \times 4\sqrt{2} - 5 \end{array}\right\}$ when $z = r$

and therefore in this caſe the ſuperficies and ſolid are algebraically aſſignable.

Becauſe $x = z\ s, a = \frac{z^3}{r^2}$, and $y = z\ c\ s, a = \sqrt{z^2 - \frac{z^6}{r^4}}$ it follows that

$x^2\ r^4 = \overline{y^2 + x^2}\,|^3$ is the equation to the curve in this caſe. This curve will paſs through B, becauſe when $z = r$, $s, a =$ radius.